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PERIODICITIES IN THE THEORY OF PARTITIONS.

BY E. T. BELL.

I. Definitions and Fundamental Identities.

1. Introduction. If in the analysis of partitions we use theta functions, not merely theta constants as customary, we enter an extensive region of the theory of partitions which has not been worked hitherto, and which abounds in remarkable relations between Euler's $P(n)$, the total number of partitions of n, and six other denumerants introduced naturally by this extension. There is great flexibility in the method owing to the presence of continuous variables in the fundamental identities which are finite expressions containing denumerants. This is in sharp contrast to the usual analysis in which all relations between denumerants and functions of a continuous parameter involve infinite series or products.

The relations between denumerants fall into classes according to the periodicity of the coefficients. Two kinds of periodicity are encountered; the first is that familiar in analysis, the second is of the kind peculiar to the theory of numbers in which functions whose arguments differ by positive integral multiples of the period are not equal but are congruent with respect to certain moduli. Let us call the first of these ordinary periodicity and the second numerical. The type of numerical periodicity occurring in connection with partitions is that which appears in the integers defined by recurring series having scales of relation with integral coefficients the first of which is unity, or, what is equivalent, the algebraic numbers associated with the series are algebraic integers. This type was noticed by Lagrange, considered in detail for second order series by Lucas, and recently extended to the general case by Carmichael*.

In one class of relations some of the coefficients have ordinary periodicity and the rest are aperiodic; in the second only numerical periodicity appears; in a third some of the coefficients exhibit ordinary periodicity and the rest numerical, so that in a sense this class is doubly periodic; in a fourth class all of the coefficients are aperiodic. On account of their greater simplicity and interest we treat only linear relations; the non-linear are given by a parallel development starting from the addition theorems for the theta functions.

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* Quarterly Journal, vol. 48 (1920), pp. 343–372.

The entire subject is interconnected with that of Lucas' functions u_n, v_n and with cyclotomy, so that we have here a link between the division of the circle into equal parts and the theory of partitions of numbers. The division of the circle has already entered indirectly into partitions through Sylvester's theory of waves. The present application is distinct from this. It will be evident also that the subject is closely related to the transformation theory of the elliptic theta functions, particularly to that part which gives the formulas for the division of the argument; but we do not discuss this.

2. *A*-, *E*-, *O*-**partitions.** We consider those partitions of the positive integer n in each of which precisely r parts occur only once and no part appears more than twice, designating the number of such partitions by $A_r(n)$ if the parts may be odd or even, by $E_r(n)$ if all the parts are restricted to be even, and by $O_r(n)$ if all the parts are restricted to be odd; r is called the rank of each partition enumerated in $A_r(n)$, $E_r(n)$ or $O_r(n)$. The total number of partitions of n is denoted by $P(n)$, and the whole number of partitions of n into parts none of which appears more than twice is written $A(n)$ if the parts may be odd or even, $E(n)$ if all the parts are even, $O(n)$ if all the parts are odd.

Hence

$$A(n) = \sum A_r(n), \qquad E(n) = \sum E_r(n), \qquad O(n) = \sum O_r(n),$$

the summation extending to $r = 0, 1, 2, \cdots$. Clearly r can not exceed $1 + \sqrt{n}$.

By convention the value of each of the seven denumerants $A_r(n)$, $A(n), \cdots$, $O(n)$, $P(n)$ is 1 when $n = 0$.

From the definitions

$$E_r(2n+1) = O_{2r}(2n+1) = O_{2r+1}(2n) = 0,$$

and it is easily seen (at once from the generators in § 3, or from first principles) that $E_r(2n) = A_r(n)$. It is therefore unnecessary although sometimes convenient to retain both the A and E functions. We need also the following in which x is a parameter,

$$A'_n(x) \equiv \sum x^r A_r(n), \qquad E'_n(x) \equiv \sum x^r E_r(n), \qquad O'_n(x) \equiv \sum x^r O_r(n),$$

the $\sum$ as before referring to $r = 0, 1, 2, \cdots$.

By convention $A'_0(x) = E'_0(x) = O'_0(x) = 1$, and from the definitions

$$A'_n(1) = A(n), \qquad E'_n(1) = E(n), \qquad O'_n(1) = O(n).$$

Of the seven functions $P(n), \cdots, O(n)$, only $A(n)$ and Euler's $P(n)$ seem to have been defined previously[*].

$A(n)$ was introduced, discussed, and generalized by Glaisher[†].

The remaining five can also be generalized in several ways immediately suggested by the forms of their generators, but as these extensions have no simple connection with the elliptic theta functions we leave them aside.

3. **Generators.** Indicate that the part α occurs precisely a times in a given partition by writing α_a. Then if $\alpha_a, \beta_b, \cdots, \gamma_c$ is a partition of n,

$$a\alpha + b\beta + \cdots + c\gamma = n,$$

and $\alpha, \beta, \cdots, \gamma$ are all different. If no part is to appear more than twice, each of $a, b, \cdots, c$ is one of the numbers 0, 1, 2. We may write $a \times n = n_a$, and hence

$$1 + xq^n + q^{2n} \equiv q^{n_0} + xq^{n_1} + q^{n_2};$$

whence the generators

$$\prod (1 + xq^n + q^{2n}) = \sum q^n A'_n(x), \tag{1}$$

$$\prod (1 + xq^l + q^{2l}) = \sum q^n E'_n(x) = \sum q^{2n} A'_n(x), \tag{2}$$

$$\prod (1 + xq^m + q^{2m}) = \sum q^n O'_n(x), \tag{3}$$

in which the respective products extend to all integers $n > 0$, to all even integers $l > 0$, to all odd integers $m > 0$, and the summations to all integers $n \geqq 0$. As obviously is permissible it is assumed that x, q both different from zero are such as to render the series and products absolutely convergent.

4. **Notation.** For brevity $l, \lambda, m, \mu, n, \nu$ unless otherwise indicated represent integers of the following kinds: l, λ are even, m, μ odd, n, ν odd or even,

$$l \geqq 0,\ \lambda \gtreqqless 0; \qquad m > 0,\ \mu \gtrless 0; \qquad n \geqq 0,\ \nu \gtreqqless 0.$$

The a^{th} triangular number, zero being counted the first, is written t_a; $t_a = \frac{1}{2} a(a-1)$, $a > 0$. The product sign $\prod$ extends to all integers except zero of the type indicated, the summation $\sum$ to all integers of the indicated type, zero included. Departures from these conventions will be specifically noted where they apply. *When the letter r occurs under a $\sum$, the summation is with respect to r only, and refers to $r = 0, 1, 2, \cdots$; similarly for a, the summation in this case extending to $a = 1, 2, 3, \cdots$.* From the nature of the functions all such sums are finite. Sums of denumerants whose arguments are functions of λ, μ or ν are extended only to all those values of λ, μ or ν consistent with their definitions which render the arguments of the summands $\geqq 0$. The significance of any formula can be seen at once by referring to this paragraph.

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[*] Cf. Dickson, History of the Theory of Numbers, vol. 2, chap. III; MacMahon, Combinatory Analysis, vol. 2.

[†] Messenger of Mathematics, vol. 12 (1883), p. 166.

5. The fundamental identities. In the usual notation of the elliptic theta functions

$$\vartheta_3(x) = q_0 \prod (1+2q^m \cos 2x + q^{2m}) = \sum q^{\nu^2} \cos 2\nu x,$$

$$q_0 \equiv q_0(q) = \prod (1-x^l), \qquad q_0^{-1} = \sum q^{2n} P(n),$$

and hence by (3),

$$q_0^{-1}\vartheta_3(\tfrac{1}{2}x) = \sum q^n O'_n(2\cos x).$$

Recalling that $O_r(n) = 0$ when r, n are not both even or both odd we equate coefficients of like powers of q in the last identity distinguishing the cases $n = l$, $n = m$:

$$\sum 2^{2r} O_{2r}(l) \cos^{2r} x = \sum P\left(\frac{l-\lambda^2}{2}\right) \cos \lambda x, \tag{4}$$

$$\sum 2^{2r+1} O_{2r+1}(m) \cos^{2r+1} x = \sum P\left(\frac{m-\mu^2}{2}\right) \cos \mu x. \tag{5}$$

In the same way from

$$\vartheta_2(x) = 2 q_0 q^{\frac{1}{4}} \cos x \prod (1+2q^l \cos 2x + q^{2l}) = \sum q^{\frac{\mu^2}{4}} \cos \mu x,$$

we find, on dividing through by q_0 and using (2),

$$2\cos x \sum 2^r E_r\left(\frac{m-1}{4}\right) \cos^r 2x = \sum P\left(\frac{m-\mu^2}{8}\right) \cos \mu x \tag{6}$$

when and only when $m \equiv 1$ mod. 8. These three can be written also

$$\sum 2^{2r} 0_{2r}(2n) \cos^{2r} x = P(n) + 2\sum P(n-2a^2) \cos 2ax, \tag{4.1}$$

$$\sum 2^{2r+1} 0_{2r+1}(2n+1) \cos^{2r+1} x = 2\sum P(n-4t_a) \cos (2a-1)x, \tag{5.1}$$

$$\cos x \sum 2^r A_r(n) \cos^r 2x = \sum P(n-t_a) \cos (2a-1)x, \tag{6.1}$$

where in the last we have replaced $E_r(2n)$ by $A_r(n)$. Multiply (6.1) throughout by $\cos x$ and replace x by $\frac{1}{2}x$,

$$\begin{aligned} 2A_0(n) &+ \sum 2^a [A_{a-1}(n) + 2A_a(n)] \cos^a x \\ &= 2P(n) + 2\sum [P(n-t_a) + P(n-t_{a+1})] \cos ax. \end{aligned} \tag{6.2}$$

In (4.1) — (6.1) change x into $x+\frac{1}{2}\pi$,

$$(4.11)\quad \sum 2^{2r} 0_{2r}(2n)\sin^{2r}x = P(n) + 2\sum(-1)^{a} P(n-2a^{2})\cos 2ax,$$

$$(5.11)\quad \sum 2^{2r+1} 0_{2r+1}(2n+1)\sin^{2r+1}x = -2\sum(-1)^{a} P(n-4t_{a})\sin(2a-1)x,$$

$$(6.11)\quad \sin x\sum(-1)^{r} 2^{r} A_{r}(n)\cos^{r}2x = -\sum(-1)^{a} P(n-t_{a})\sin(2a-1)x;$$

and in (6.2) change x into $\pi - x$,

$$(6.21)\quad \begin{aligned} &2A_{0}(n) + \sum(-1)^{a}2^{a}[A_{a-1}(n) + 2A_{a}(n)]\cos^{a}x \\ &= 2P(n) + 2\sum(-1)^{a}[P(n-t_{a}) + P(n-t_{a+1})]\cos ax. \end{aligned}$$

6. **Introduction of Lucas' functions**[*]. It follows from De Moivre's theorem that if p, q are real and x is the real or imaginary angle such that

$$(7)\quad 2\sqrt{q}\cos x = p, \qquad 2\sqrt{q}\sin x = \sqrt{4q-p^{2}},\ 4q \neq p^{2}\dagger,$$

then for all values $\geqq 0$ of n,

$$(8)\quad 2q^{\frac{n}{2}}\cos nx = v_{n}, \qquad 2q^{\frac{n}{2}}\sin nx = u_{n}\sqrt{4q-p^{2}},$$

where u_n, v_n is that pair of linearly independent solutions of the difference equation

$$(9)\quad y_{n+2} = py_{n+1} - qy_{n}$$

which is defined by the initial values

$$(10)\quad u_{0} = 0,\ u_{1} = 1,\ v_{0} = 2,\ v_{1} = p.$$

When p, q are integers $\gtrless 0$, u_n, v_n are the singly periodic numerical functions of Lucas, the implied periodicity in this definition being numerical as defined in § 1 and independent of the period $2\pi/n$ of the circular functions in (8). For assigned values of p, q the moduli with respect to which this periodicity obtains are determinate, as also are the sequences u_n, v_n $(n = 0, 1, \ldots)$. The particular sequences defined by $p = a$, $q = b$ (a, b constants) are called the u, v sequences for (a, b). To eight of the most interesting Lucas gave the following names: $(1, -1)$, Fibonacci; $(2, -1)$, Pell; $(2, -3)$ conjugate Pell; $(3, 2)$ Fermat. As will be seen later these cases play an important in the theory of partitions. The Fibonacci u-sequence 0, 1, 1, 2, 3, 5, . . . is usually called Fibonacci's numbers, although Bachmann gives this name to the v-sequence 2, 1, 3, 4, 7, 11, . . ., calling the first Lamé's. He also names Pell's sequences after Dupré. When p, q are integers we shall call u_n, v_n $(n = 0, 1, 2, \ldots)$ the Lucas' u, v sequences for (p, q).

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[*] E. Lucas, (1) American Journal of Mathematics, vol. 1 (1878) pp. 184–238, 289–321; (2) Theorie des Nombres, chap. 18, cf. also Bachmann, Niedere Zahlentheorie, vol. 2, chap. 3; Dickson, loc. cit. vol. 1, chap. 17, also papers of Cipolla, ibid. p. 218 for properties relating to the periodicity. Note that in (2) Lucas abandons the restriction that p, q be relatively prime in the definition of u_n, v_n, while Bachmann retains it. The implications of this distinction do not concern us here; cf. Carmichael, loc. cit. § 2.

† If $4q = p^{2}$ the results are developed otherwise, cf. § 18.

In what immediately follows p, q are arbitrary real quantities. Write $\delta = \sqrt{p^2 - 4q}$, so that

$$\delta \frac{\partial \delta}{\partial p} = p, \qquad \delta \frac{\partial \delta}{\partial q} = -2,$$

and hence from (7), (8):

$$(11) \qquad \frac{\partial v_n}{\partial p} = n u_n, \qquad \frac{\partial v_n}{\partial q} = -n u_{n-1},$$

$$(12) \qquad \delta \frac{\partial (\delta u_n)}{\partial p} = n v_n, \qquad \delta \frac{\partial (\delta u_n)}{\partial q} = -n v_{n-1},$$

$$(13) \qquad \int u_n \, dp = 1/n \cdot v_n, \qquad \int u_n \, dq = -1/(n+1) \cdot v_{n+1},$$

$$(14) \qquad 1/\delta \cdot \int v_n/\delta \, dp = 1/n \cdot u_n, \qquad 1/\delta \cdot \int v_n/\delta \, dq = -1/(n+1) \cdot u_{n+1},$$

and the explicit values of u_n, v_n are given by

$$(15) \qquad 2^n \delta u_n = (p+\delta)^n - (p-\delta)^n, \qquad 2^n v_n = (p+\delta)^n + (p-\delta)^n,$$

these being a pair of fundamental solutions of (9).

7. Ordinary periodicity. This occurs in connection with partitions as follows. Let $\alpha(n)$ be a function which takes a single integral value for each value $\geqq 0$ of the integer n. Then if there exists a finite integer $n' > 0$ such that for k an arbitrary integer $\geqq 0$, $\alpha(n + kn') = \alpha(n)$ $(n = 0, 1, 2, \cdots)$, n' is a period of $\alpha(n)$; and if m is the least period of $\alpha(n)$ the values of $\alpha(n)$

recur in cycles of m and in no smaller cycle. Let us call this least cycle of values the recurrence of $\alpha(n)$ and indicate it thus,

$$\begin{aligned} n &\equiv 1, 2, 3, \cdots, m-1, 0 \bmod. m, \\ \alpha(n) &= \alpha(1), \alpha(2), \alpha(3), \cdots, \alpha(m-1), \alpha(0), \end{aligned}$$

giving in any specific case the actual values of $\alpha(1), \alpha(2), \cdots, \alpha(0)$. This kind of periodicity appears in sums of the forms

$$\sum \alpha(a) P(n - 2a^2), \qquad \sum \alpha(a) P(n - ct_a), \qquad c = 1, 4.$$

Such sums are not periodic functions of m or of n; the periodicity is only in the cyclic recurrence of the coefficients. When m is the period of $\alpha(n)$ it is convenient to say that each of these sums is of period m.

To see more clearly the nature of relations involving $\alpha(n)$ functions consider the sum of period 5,

$$\begin{aligned} -P(n-1) + 2P(n-3) - P(n-6) - P(n-21) \\ + 2P(n-28) - P(n-36) - P(n-66) - \cdots, \end{aligned}$$

which will be shown equal to

$$-A_1(n) + A_2(n) - 2A_3(n) + 3A_4(n) - 5A_5(n) + 8A_6(n) - 13A_7(n) + \cdots,$$

the coefficients $1, 1, 2, 3, 5, 8, \cdots$ being Fibonacci's u. The arguments of the P's are of the form $n - t_a$ and decrease; the argument n of the A's is constant, and the rank variable. Thus the periodic (ordinary) sum is reduced to a linear function of A-partitions of n alone, the coefficients in which are numerically periodic. The relation can be written

$$\sum \alpha(a) P(n - t_a) = \sum (-1)^r u_r A_r(n),$$

where u_r is the r^{th} Fibonacci number, and the recurrence of $\alpha(n)$ is

$$\begin{aligned} a &\equiv 1, \quad 0, \ 3, \quad 4, \ 0, \ \bmod. 5, \\ \alpha(a) &= 0, -1, \ 2, -1, \ 0. \end{aligned}$$

Fibonacci's u_{r+2} has already appeared in the theory of partitions, MacMahon (loc. cit. p. 46) having introduced it from other considerations to enumerate the whole number of partitions of all numbers having no sequences, no repetitions and no part greater than r. It may be possible therefore to interpret the foregoing sum of A-functions in terms of partitions of n of one particular kind. The like applies to sums involving binomial coefficients (§ 18), for the latter also have assigned meanings as denumerants by MacMahon (ibid. p. 48).

II. Relations in which Lucas' Numerically Periodic Coefficients Occur.

8. **General relations.** The u_n, v_n in (I)-(VIII) are any solutions of (9) determined by the initial conditions (10). Substitute in (4.1)-(6.21) the values of $\cos x$, $\cos ax$ from (7), (8); to get (IV), (VIII) add and subtract the pair thus found from (6.2), (6.21); write out the others directly from the substitution:

$$\text{(I)} \qquad \sum (p^2/p)^r O_{2r}(2n) = P(n) + \sum \frac{v_{2a}}{q^a} P(n - 2a^2),$$

$$\text{(II)} \qquad p \sum (p^2/q)^r O_{2r+1}(2n+1) = q \sum \frac{v_{2a-1}}{q^a} P(n - 4t_a),$$

$$\text{(III)} \qquad p \sum (p^2/q - 2)^r A_r(n) = q \sum \frac{v_{2a-1}}{q^a} P(n - t_a),$$

$$\text{(IV)} \qquad 2A_0(n) + \sum (p^2/q)^a [A_{2a-1}(n) + 2A_{2a}(n)] \\ = 2P(n) + \sum \frac{v_{2a}}{q^a} [P(n - t_{2a}) + P(n - t_{2a+1})],$$

$$\text{(V)} \qquad \sum (4 - p^2/q)^r O_{2r}(2n) = P(n) + \sum (-1)^a \frac{v_{2a}}{q^a} P(n - 2a^2),$$

$$\text{(VI)} \qquad \sum (4 - p^2/q)^r O_{2r+1}(2n+1) = -q \sum (-1)^a \frac{u_{2a-1}}{q^a} P(n - 4t_a),$$

$$\text{(VII)} \qquad \sum (2 - p^2/q)^r A_r(n) = -q \sum (-1)^a \frac{u_{2a-1}}{q^a} P(n - t_a),$$

$$\text{(VIII)} \qquad \sum (p^2/q)^a [A_{2a-2}(n) + 2A_{2a-1}(n)] \\ = \sum \frac{v_{2a-1}}{q^a} [P(n - t_{2a-1}) + P(n - t_{2a})].$$

Each of these generates an infinite chain of results of the same general type upon repeated application of (11)–(14) in any order. At the n^{th} step there are possible $8 \times 7^{n-1}$ different relations deduced from any one. For at each step may be applied any one of the eight processes in (11)–(14) except the inverse of that last used. Thus the p-derivative of (I) is

$$\sum r (p^2/q)^r O_{2r}(2n) = p \sum \frac{a u_{2a}}{q^a} P(n - 2a^2);$$

multiply this by δ, take again the p-derivative and multiply by δ,

$$\sum[r^2(p^2/q)^r + 2r(2r-1)(p^2/q)^{r-1}]\,O_{2r}(2n) = \sum\frac{a^2 v_{2a}}{q^a}P(n-2a^2),$$

and so on.

9. **Relations with Lucas' Functions.** Thus far p, q are arbitrary real quantities. When p, q are integers, the u, v in all of the above relations are Lucas' functions. For the specially named sequences in § 6 the results are particularly simple and interesting, but we need not write them out.

In the case of arbitrary p, q it is clear that in any of the relations coefficients of p^a, q^b can be equated after u, v have been replaced by their explicit forms (15) and expanded. The result is a chain of aperiodic relations. Some of the more important of these are obtained otherwise in § 18.

III. Ordinary Periodicity of Odd Prime Period p.

10. **Preliminary reductions of cyclotomic sums.** This class of relations depends upon the $e = \frac{1}{2}(p-1)$ binomial periods

$$\eta'_k = r^{g^k} + r^{-g^k} = 2\cos g^k\theta_p \, (0 \leqq k < e)$$

appertaining to the p-section of the circle*, p an odd prime, $\theta_p = 2\pi/p$, $g =$ a primitive root of p. In what follows it is immaterial which g be selected, but the choice once made is to be permanent, and likewise for the period η chosen presently. As j runs through all integers $\geqq 0$, η'_j takes only the e distinct values $\eta'_k(0 \leqq k < e)$.

The values of $2\cos a\theta_p(0 < a \leqq e)$ are all different, and if a' is the least positive index of a to the base g,

$$\eta'_{a'} = 2\cos a\theta_p = 2\cos(p-a)\theta_p. \qquad (0 < a \leqq e)$$

It follows that $2\cos a\theta_p(0 < a \leqq e)$ is a permutation of $\eta'_k(0 \leqq k < e)$. Write $\eta'_{a'} = \eta_a$, so that the e binomial periods are

$$\eta_a = 2\cos a\theta_p = 2\cos(p-a)\theta_p. \qquad (0 < a \leqq e)$$

By convention η_0 (not one of the periods) has the value 2. Reduced forms of four sums will be required.

Let η be a particular one of the binomial periods and $\lambda_r(n)(r = 0, 1, 2, \cdots)$ single valued functions real for all values $\geqq 0$ of the integer n.

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* Cf. Mathews, Theory of Numbers, chap. 7; Bachmann, Kreistheilung, chap. 6. There is no corresponding class based on the f-nomial periods when $f > 2$.

Then

$$\lambda(n) = \sum \eta^r \lambda_r(n) \tag{16}$$

is uniquely reducible to

$$\lambda(n) = \lambda_0(n) + L_1(n)\eta_1 + L_2(n)\eta_2 + \cdots + L_e(n)\eta_e, \tag{17}$$

in which $L_s(n)(0 < s \leqq e)$ are real. The appropriate reduction of $L_s(n)$ being rather long we first state the result for clearness. Since η is a *particular* period, one and only one set of $e(e-1)$ integers c_{jk} is defined by the identities

$$\eta^j = c_{j1}\eta_1 + c_{j2}\eta_2 + \cdots + c_{je}\eta_e \qquad (0 < j < e) \tag{18}$$

obtained by expressing each of $\eta, \eta^2, \cdots, \eta^{e-1}$ as a linear homogeneous function of all the periods; moreover the e periods are the roots of an irreducible equation

$$\eta^e = a_{e-1}\eta^{e-1} + a_{e-2}\eta^{e-2} + \cdots + a_1\eta + a_0 \tag{19}$$

with integral coefficients. The e^2 integers

$$c_{js}, a_k (0 < j < e, 0 < s \leqq e, 0 \leqq k < e), \tag{20}$$

together with the recurrence relation of order e

$$f(n+e) = a_{e-1}f(n+e-1) + a_{e-2}f(n+e-2) + \cdots + a_0 f(n), \tag{21}$$

whose associated algebraic equation is the equation (19) of the periods, determine $L_s(n)$ for all values $\geqq 0$ of the integer n. For it will be shown that

$$L_s(n) = \sum \lambda_a(n) u_s(a), \qquad (0 < s \leqq e) \tag{22}$$

the summation by § 4 being with respect to $a = 1, 2, 3, \cdots$, where $u_s(n)(n \geqq 0)$ is that solution of (21) which is given by the $e-1$ initial values

$$u_s(j) = c_{js} (0 < j < e). \tag{23}$$

To prove this reduce (16) by (19) to

$$\lambda(n) = \lambda_0(n) + A_1(n)\eta + A_2(n)\eta^2 + \cdots + A_{e-1}(n)\eta^{e-1}, \tag{24}$$

in which the A's and λ_0 are independent of η, and write

$$\eta^n = \varphi_{e-1}(n)\eta^{e-1} + \varphi_{e-2}(n)\eta^{e-2} + \cdots + \varphi_0(n), \tag{25}$$

the φ's being integers. Multiply (25) throughout by η, reduce the right of this result by the right of (19), observe that by (25) the left of the identity so obtained is

$$\varphi_{e-1}(n+1)\eta^{e-1}+\varphi_{e-2}(n+1)\eta^{e-2}+\cdots+\varphi_0(n+1),$$

and equate coefficients of like powers of η, getting

$$(26)\quad \varphi_0(n+1)=a_0\varphi_{e-1}(n),\quad \varphi_j(n+1)=a_j\varphi_{e-1}(n)+\varphi_{j-1}(n).\quad (0<j<e)$$

From (26) it follows that

$$(27)\qquad \varphi_0(n),\quad \varphi_1(n),\cdots,\quad \varphi_{e-1}(n)$$

is that set of e linearly independent solutions of (21) which is defined by the system of initial values

$$(28)\quad \varphi_j(j)=1\ (0\leqq j<e),\quad \varphi_j(k)=0\ (0\leqq j<e,\quad k<e,\quad j\neq k);$$

that is, (27) is the fundamental set of solutions of (21). Substitute in (16) the values (25) of η, η^2, $\cdots$ and compare with (24):

$$(29)\qquad A_j(n)=\sum\lambda_a(n)\,\varphi_j(a),\qquad (0<j<e)$$

the $\sum$ by § 4 referring to $a=1,2,3,\cdots$. In (24) substitute for $A_j(n)$ from (29), η^j $(j=1,2,\cdots,e-1)$ from (18), and compare with (17). The coefficient of $\lambda_a(n)$ in $L_s(n)$, that is $u_s(a)$, is

$$(30)\qquad u_s(a)=\sum_{j=1}^{e-1}c_{js}\,\varphi_j(a);$$

whence by comparing (30), (28) we have the stated initial conditions (23), which completes the verification of (22).

With this is a complementary reduction obtained from the preceding by changing the sign of each period. The constants (20) are as before, and we find immediately that

$$(16.1)\qquad \mu(n)=\sum(-1)\,\eta^r\mu_r(n)=\mu_0(n)+\sum(-1)^a\eta^a\mu_a(n)$$

can be reduced to

$$(17.1)\qquad \mu(n)=\mu_0(n)-M_1(n)\,\eta_1-M_2(n)\,\eta_2-\cdots-M_e(n)\,\eta_e,$$

where now

$$(22.1)\qquad M_s(n)=\sum\mu_a(n)\,v_s(a),\qquad (0<s\leqq e)$$

and $v_s(n)$ is that solution of

$$h(n+e)=\sum_{j=1}^{e}(-1)^j a_{e-j}h(n+e-j), \tag{21.1}$$

which is defined by the $e-1$ initial values

$$v_s(j)=(-1)^{j-1}c_{js}; \qquad (0<j<e) \tag{23.1}$$

the formulas being numbered correspondingly to those in the first reduction.

The next type of sum, depending on the values of $2\cos a\theta_p$ $(a=1,2,3,\cdots)$, introduces the $p+1$ elementary periodic functions upon which the relations of period p are based. Write $[x]$ for the greatest integer not greater than x, hence $e=[\frac{1}{2}p]$, and define $e+1$ periodic functions $\alpha_k(a)$ $(0\leqq k\leqq e)$ by

$$\begin{aligned} \alpha_{2j}(a)&=1, & a&\equiv\pm j \bmod. p, & 0&<j\leqq[\tfrac{1}{4}p]; \\ \alpha_{p-2j}(a)&=1, & a&\equiv\pm j \bmod. p, & [\tfrac{1}{4}p]&<j\leqq[\tfrac{1}{2}p]; \\ \alpha_0(a)&=1, & a&\equiv 0 \bmod. p, \end{aligned} \tag{31}$$

the values of $\alpha_k(a)$ in all cases except these being zero. Then it is easy to verify that

$$2\cos 2a\theta_p=\alpha_k(a)\,\eta_k \qquad (0\leqq k\leqq e) \tag{32}$$

for all values >0 of the integer a. We recall that η_0 has the conventional value 2. In the same way $e+1$ periodic functions $\beta_k(a)$ $(0\leqq k\leqq e)$ are defined by

$$\begin{aligned} \beta_{2j-1}(a)&=1, & a&\equiv j,\,1-j \bmod. p, & 0&<j\leqq[\tfrac{1}{4}(p+1)]; \\ \beta_{p-2j+1}(a)&=1, & a&\equiv j,\,1-j \bmod. p, & [\tfrac{1}{4}(p+1)]&<j\leqq[\tfrac{1}{2}p]; \\ \beta_0(a)&=1, & a&\equiv \tfrac{1}{2}(p+1) \bmod. p, \end{aligned} \tag{33}$$

the values of $\beta_k(a)$ in all other cases being zero; and for all values of the integer $a>0$,

$$2\cos(2a-1)\,\theta_p=\beta_k(a)\eta_k. \qquad (0\leqq k\leqq e) \tag{34}$$

Let ψ_0, χ_0 be real quantities, $\psi(a)$, $\chi(a)$ functions which are single valued and real for each integer $a>0$. Then by (32) the sum

$$\psi_0+2\sum\psi(a)\cos 2a\theta_p \tag{35}$$

is reduced to

$$\Psi_0+\Psi_1\,\eta_1+\Psi_2\,\eta_2+\cdots+\Psi_e\,\eta_e, \tag{36}$$

in which the Ψ's are real, and

$$(37)\qquad \Psi_0 = \psi_0 + 2\sum\alpha_0(a)\,\psi(a), \quad \Psi_j(a) = \sum\alpha_j(a)\,\psi(a). \qquad (0<j\leqq e)$$

Similarly by (34),

$$(38)\qquad \chi_0 + 2\sum\chi_a\cos(2a-1)\,\theta_p = X_0 + \sum_{j=1}^{e}X_j\eta_j,$$

$$(39)\qquad X_0 = \chi_0 + 2\sum\beta_0(a)\,\chi(a), \quad X_j = \sum\beta_j(a)\,\chi(a). \qquad (0<j\leqq e)$$

11. **Systems of** e **relations each of odd prime ordinary period** p. Take $\eta = \eta_1$ in (18), *thus fixing the integers* c_{js} *in* (20). These values are understood in the rest of this section. The a_k in (20) are the same for all choices of η. In each of (IX)-(XII) below j is defined by $0<j\leqq e$, so that each equality represents $e=\frac{1}{2}(p-1)$ relations in which the functions α', β', γ' are ordinary periodic of odd prime period p, and by Carmichael's work cited in § 1, the functions u, v have numerical periodicity.

In (4.1) put $x=\theta_p$, hence $2\cos x = \eta_1$, and reduce the result by (17), (36) and $\sum_{j=1}^{e}\eta_j = -1$ to

$$\sum_{j=1}^{e}\left[-O_0(2n) + \sum u_j(2a)\,O_{2a}(2n)\right]\eta_j$$
$$= \sum_{j=1}^{e}\left[-P(n) + \sum\{\alpha_j(a) - 2\alpha_0(a)\}\,P(n-2a^2)\right]\eta_j.$$

Write $\alpha_j(a) - 2\alpha_0(a) = \alpha'_j(a)$ and in the preceding equate coefficients of η_j:

$$(\mathrm{IX})\qquad O_0(2n) - \sum u_j(2a)\,O_{2a}(2n) = P(n) - \sum\alpha'_j(a)\,P(n-2a^2),$$

in which, from (31), $\alpha'_k(a) = 0$ $(0<k\leqq e)$ except in the cases

$$(40)\qquad \begin{array}{lll} a\equiv\pm j\ \mathrm{mod.}\,p, & 0<j\leqq[\frac{1}{4}p], & \alpha'_{2j}(a)=1, \\ a\equiv\pm j\ \mathrm{mod.}\,p, & [\frac{1}{4}p]<j\leqq[\frac{1}{2}p]. & \alpha'_{p-2j}(a)=1, \\ a\equiv 0\ \mathrm{mod.}\,p, & 0<j\leqq[\frac{1}{2}p], & \alpha'_j(a)=-2. \end{array}$$

The u-coefficients are integers determined by (21), (23).

For the same u's, (5.1) compared with (37)-(39) yields the system

$$(\mathrm{X})\qquad \sum u_j(2a-1)\,O_{2a-1}(2n+1) = \sum\beta'_j(a)\,P(n-4t_a);$$

$\beta_k(a) = 0 (0 < k \leqq e)$ except in the cases

$$
(41)\quad \begin{array}{lll}
a \equiv j,\ 1 - j \bmod. p, & 0 < j \leqq [\tfrac{1}{4}(p+1)], & \beta'_{2j-1}(a) = 1, \\
a \equiv j,\ 1 - j \bmod. p, & [\tfrac{1}{4}(p+1)] < j \leqq [\tfrac{1}{2}p], & \beta'_{p-2j+1}(a) = 1, \\
a \equiv \tfrac{1}{2}(p+1) \bmod. p, & 0 < j \leqq [\tfrac{1}{2}p], & \beta'_j(a) = -2.
\end{array}
$$

In (6.2) put $x = \theta_p$ and get for the same u's the system

$$
\begin{aligned}
\text{(XI)}\quad 2A_0(n) - \sum u_j(a)[A_{a-1}(n) + 2A_a(n)] \\
= 2P(n) - \sum \gamma'_j(a)[P(n - t_a) + P(n - t_{a+1})];
\end{aligned}
$$

$\gamma'_k(a) = 0 (0 < k \leqq e)$ except in the cases

$$
(42)\quad \begin{array}{lll}
a \equiv \pm j \bmod. p, & 0 < j \leqq [\tfrac{1}{2}p], & \gamma'_j(a) = 1; \\
a \equiv 0 \bmod. p, & 0 < j \leqq [\tfrac{1}{2}p], & \gamma'_j(a) = -2.
\end{array}
$$

Comparing (6.21) with (16.1) we find for the same γ' the system

$$
\begin{aligned}
\text{(XII)}\quad 2A_0(n) + \sum v_j(a)[A_{a-1}(n) + 2A_a(n)] \\
= 2P(n) - \sum (-1)^a \gamma'_j(a)[P(n - t_a) + P(n - t_{a+1})],
\end{aligned}
$$

where the v's are integers determined by (21.1), (23.1).

13. The recurrence (§ 7) of each of the periodic functions $\alpha'_j(a)$, $\beta'_j(a)$, $\gamma'_j(a)$ consists of p terms one of which is -2 and of the remaining $p-1$ terms two are units $(+1)$ and the rest zeros. In the recurrences of $\alpha'_j(a)$, $\gamma'_j(a)$ the terms, the last which is -2 being omitted, read the same backwards and forwards; the recurrence of $\beta'_j(a)$ is symmetrical about its middle term -2. Each of the systems (IX)-(XII) determines a relation in which the recurrence for the periodic function consists of $[\frac{1}{2}p]$ arbitrary constants k_j and retains its symmetry as above. In these the coefficients u or v on the left are again given by the recurring series (21) or (21.1) but with sets of $e-1$ initial values depending linearly on the k_j. If the constants k_j are integers the u, v are, by Carmichael's work as before, numerically periodic. It will be sufficient to write out the relation of this type deduced from (IX); the remaining three are found in the same way.

In (IX) replace j by $2j$, multiply by k_{2j} and (glancing at (40)) put in turn for j each of the integers $0 < j \leqq [\frac{1}{4}p]$, in (IX) replace j by $p - 2j$ and let j range $[\frac{1}{4}p] < j \leqq [\frac{1}{2}p]$; add all the relations thus obtained. Write $k_0 = 2(k_1 + k_2 + \cdots + k_e)$. Then if $\alpha(a)$ is the periodic function whose recurrence is

$$
(43)\quad \begin{array}{l}
a \equiv 1, 2, 3, \cdots, p-3, p-2, p-1, 0 \bmod. p, \\
\alpha(a) = k_1, k_2, k_3, \cdots, k_3, k_2, k_1, -2k_0,
\end{array}
$$

and $u(r)$ $(r = 1, 2, 3, \cdots)$ is that solution of (21) which is determined by the initial values

$$u(r) = \sum_{j=1}^{e} k_j c_{rj} \quad (0 < r < e), \tag{44}$$

we have for the sum just constructed

$$k_0 O_0(2n) - \sum u(2a) O_{2a}(2n) = k_0 P(n) - \sum \alpha(a) P(n - 2a^2). \tag{XIII}$$

IV. Relations Involving the Numbers of Fibonacci, Pell, Fermat and Lucas.

14. **Dependence upon quadratic integers.** In section II the Lucas numbers appear as coefficients of the P functions; in the class to be considered next these numbers are coefficients of the A, O functions and the coefficients of the P's are ordinary-periodic. Let d, p_0, q_0, p_1, q_1 be rational integers, of which $d > 1$ has no square divisor > 1; p_0, q_0 are arbitrary, and $p_1 \equiv q_1 \bmod. 2$. All integers ζ_0, ζ_1 of the corpus $K(\sqrt{d})$ fall into the classes*

$$d \equiv 2, 3 \bmod. 4, \qquad \zeta_0 = p_0 + q_0 \sqrt{d}; \tag{45}$$

$$d \equiv 1 \bmod. 4, \qquad \zeta_1 = \frac{p_1 + q_1 \sqrt{d}}{2}. \tag{46}$$

Let $u_n, v_n (n = 0, 1, \cdots)$ be the Lucas sequences (§ 6) for

$$(p, q) = (2p_0, p_0^2 - dp_0^2); \tag{45.1}$$

then it follows that

$$2\zeta_0^n = v_n + q_0 \sqrt{d} u_n; \tag{45.2}$$

and if $u_n, v_n (n = 0, 1, \cdots)$ are the Lucas sequences for

$$(p, q) = \left(p_1, \frac{p_1^2 - dq_1^2}{2}\right), \tag{46.1}$$

then

$$2\zeta_1^n = v_n + q_1 \sqrt{d} u_n. \tag{46.2}$$

Considering first the primitive intergers of $K(\sqrt{d})$ we take $q_0, q_1 \neq 0$, and find in what cases $a \cos^c b\varphi_k$, $a \sin^c b\varphi_k$ are quadratic integers, a, b, c, k being rational integers > 0 and $\varphi_k = \pi / k$. If q_0 or $q_1 < 0$, we take $-\zeta_0$ or $-\zeta_1$

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* Cf. Bachmann, Allgemeine Arithmetik, chap. 5; Reid, Algebraic Numbers, chap. 10.

instead of ζ_0 or ζ_1 in (45.2), (46.2); and if $p_0 = d$ or $p_1 = d$ we take $\zeta_0/\sqrt{d}$, $\zeta_1/\sqrt{d}$ after having made where necessary the foregoing change of sign. These changes give the essentially simplest Lucas sequences; otherwise we get finally a different but equivalent set of relations between denumerants*.

We find the following eight cases, classified according to the sequences involved.

$(p, q) = (1, -1)$ *Fibonacci* u, v:

$$\begin{aligned} 2^{2n+1}\cos^{2n}\varphi_{10} &= 2^{2n+1}\sin^{2n}2\varphi_5 = (\sqrt{5})^n(v_n+\sqrt{5}u_n);\\ 2^{n+1}\cos^{n}\varphi_{5} &= 2^{n+1}\sin^{n}3\varphi_{10} = v_n+\sqrt{5}u_n;\\ 2^{2n+1}\cos^{2n}3\varphi_{10} &= 2^{2n+1}\sin^{2n}\varphi_5 = (-\sqrt{5})^n(v_n-\sqrt{5}u_n);\\ 2^{n+1}\cos^{n}2\varphi_{5} &= 2^{n+1}\sin^{n}\varphi_{10} = (-1)^n(v_n-\sqrt{5}u_n). \end{aligned} \tag{47}$$

$(p, q) = (2, -1)$; *Pell* u, v:

$$\begin{aligned} 2^{2n+1}\cos^{2n}\varphi_{8} &= 2^{2n+1}\sin^{2n}3\varphi_8 = (\sqrt{2})^n(v_n+\sqrt{2}u_n);\\ 2^{2n+1}\cos^{2n}3\varphi_{8} &= 2^{2n+1}\sin^{2n}\varphi_8 = (-\sqrt{2})^n(v_n-\sqrt{2}u_n). \end{aligned} \tag{48}$$

$(p, q) = (4, 1)$; *Lucas* u, v:

$$\begin{aligned} 2^{2n+1}\cos^{2n}\varphi_{12} &= 2^{2n+1}\sin^{2n}5\varphi_{12} = v_n+\sqrt{3}u_n;\\ 2^{2n+1}\cos^{2n}5\varphi_{12} &= 2^{2n+1}\sin^{2n}\varphi_{12} = v_n-\sqrt{3}u_n. \end{aligned} \tag{49}$$

The last case, $(p, q) = (4, 1)$, has been named after Lucas for convenience. These three cases correspond to the division of the circle into 10 or 20, 16, 24 parts; and for the divisions into 2, 4, 6, 8, 12 parts we have $\cos^n\varphi_1 = (-1)^n$, $\sin^n\varphi_2 = 1$;

$$\begin{aligned} 2^n\cos^n\varphi_3 &= 2^n\sin^n\varphi_6 = 1;\\ 2^n\cos^n\varphi_4 &= 2^n\sin^n\varphi_4 = (\sqrt{2})^n;\\ 2^n\cos^n\varphi_6 &= 2^n\sin^n\varphi_3 = (\sqrt{3})^n. \end{aligned} \tag{50}$$

In the Fermat case $(p, q) = (3, 2)$ $u_n = 2^n - 1$, $v_n = 2^n + 1$;
$(p, q) = (3, 2)$, *Fermat* u, v:

$$\begin{aligned} 2^n\left[\cos^n\varphi_1 \mp (-1)^n\cos^n\varphi_3\right] &= (-1)^n u_n,\quad (-1)^n v_n;\\ 2^n(\sin^n\varphi_2 \mp \sin^n\varphi_6) &= u_n,\ v_n. \end{aligned} \tag{51}$$

---

* In checking the next list the table in Hobson's Trigonometry (Cambridge, 1897), p. 72 will be found useful.

Recapitulating: when $k = 1, 2, 3, 4, 5, 6, 8, 10, 12$, we can choose rational integers $a, b, c, > 0$ so that $a \cos^c b\varphi_k$, $a \sin^c b\varphi_k$ are quadratic integers, and the values of these functions then are given by (47) — (51).

The generalization to the case of algebraic integers of degree > 2 is sufficiently obvious. We do not consider this now, as clearly the associated u, v are not Lucas' sequences.

15. **Periodic functions.** To have readily accessible the data for a complete discussion of the relations in which the associated algebraic integers are of the second degree we assemble here all the ordinary periodic functions occurring first. From these, which form a sort of fundamental set, all remaining periodic functions belonging to this case are constructed as in the derivations in § 16. To save space we adopt an abridged notation. Write the elements in the recurrence of a periodic function $\alpha(a)$ in the order in which they occur, beginning with the element which gives the value of the function when the argument is congruent to 1 for the given modulus m. If any element, $-a$, is negative, write the minus sign above it, $\bar{a}$; and if an element $\bar{a}$ (or b) is repeated j times in succession, write it $\bar{a}_j$ (or b_j), and similarly for repeated sequences of elements; thus $(ab)_j$ means $ababab\cdots$ to j terms ab. Finally enclose the result in $(\;)_m$the suffix indicating the modulus, and write $\alpha(a) = (*)_m$, the $*$ denoting the elements written as above described. For example the recurrence

$$\begin{aligned} a &\equiv 1, 2, 3, 4, \quad 5, \quad 6, 7, 8, 9, 0 \text{ mod. } 10, \\ \alpha(a) &= 1, 0, 0, 0, -1, -1, 0, 0, 0, 1 \end{aligned}$$

is abridged to $\alpha(a) = (1\, 0_3\, \bar{1}_2\, 0_3\, 1)_{10}$. The following can be verified by inspection.

$$\begin{aligned} 4\cos a\varphi_5 &= \alpha_1(a) + \alpha_1'(a)\sqrt{5}, \\ 4\cos 2a\varphi_5 &= \alpha_2(a) + \alpha_2'(a)\sqrt{5}, \\ 4\cos(2a-1)\varphi_5 &= \alpha_3(a) + \alpha_3'(a)\sqrt{5}, \\ \sin(2a-1)\varphi_5 &= \alpha_4(a)\sin\varphi_5 + \alpha_4'(a)\sin 2\varphi_5: \end{aligned} \tag{47.1}$$

$$\begin{aligned} \alpha_1(a) &= ((1\,\bar{1})_2\,\bar{4}\,(\bar{1}\,1)_2\,4)_{10}, & \alpha_1'(a) &= (1_2\,\bar{1}_2\,0\,\bar{1}_2\,1_2\,0)_{10}, \\ \alpha_2(a) &= (\bar{1}_4\,4)_5, & \alpha_2'(a) &= (1\,\bar{1}_2\,1\,0)_5, \\ \alpha_3(a) &= (1_2\,\bar{4}\,1_2)_5, & \alpha_3'(a) &= (1\,\bar{1}\,0\,\bar{1}\,1)_5, \\ \alpha_4(a) &= (1\,0_3\,\bar{1})_5, & \alpha_4'(a) &= (0\,1\,0\,\bar{1}\,0)_5. \end{aligned}$$

$$\begin{aligned} 4\cos a\varphi_{10} &= \alpha_5(a) + \alpha_5'(a)\sqrt{5} + \alpha_5''(a)\cos\varphi_{10} + \alpha_5'''(a)\cos 3\varphi_{10}, \\ \cos(2a-1)\varphi_{10} &= \alpha_6(a)\cos\varphi_{10} + \alpha_6'(a)\cos 3\varphi_{10}, \\ 4\sin(2a-1)\varphi_{10} &= \alpha_7(a) + \alpha_7'(a)\sqrt{5}: \end{aligned} \tag{47.2}$$

$$\alpha_5(a) = ((0\ 1\ 0\ \bar{1})_2\ 0\ \bar{4}\ (0\ \bar{1}\ 0\ 1)_2\ 0\ 4)_{20}, \quad \alpha_5'(a) = (0\ (1\ 0)_2\ (\bar{1}\ 0)_2\ 0_2\ (\bar{1}\ 0)_2\ (1\ 0)_2\ 0)_{20},$$
$$\alpha_5''(a) = (4\ 0_7\ \bar{4}\ 0\ \bar{4}\ 0_7\ 4\ 0)_{20}, \quad \alpha_5'''(a) = (0_2\ 4\ 0_3\ \bar{4}\ 0_5\ \bar{4}\ 0_3\ 4\ 0_3)_{20},$$
$$\alpha_6(a) = (1\ 0_3\ \bar{1}_2\ 0_3\ 1)_{10}, \quad \alpha_6'(a) = (0\ 1\ 0\ \bar{1}\ 0_2\ \bar{1}\ 0\ 1\ 0)_{10},$$
$$\alpha_7(a) = (\bar{1}\ 1\ 4\ (1\ \bar{1})_2\ \bar{4}\ \bar{1}\ 1)_{10}, \quad \alpha_7'(a) = (1_2\ 0\ 1_2\ \bar{1}_2\ 0\ \bar{1}_2)_{10}.$$

It is important to observe that $K(\sqrt{5})$, $K(\cos\varphi_{10})$, $K(\cos 3\varphi_{10})$ are all different, and hence in a linear relation with rational coefficients between $\sqrt{5}$, $\cos\varphi_{10}$, $\cos 3\varphi_{10}$, coefficients of these, and the rational terms, may be separately equated. Similarly for $K(\sin\varphi_5)$, $K(\sin 2\varphi_5)$. For the reduction of non-linear functions of these irrationals the following will he found sufficient,

$$\sqrt{5}\cos\varphi_{10} = 2\cos 3\varphi_{10} + \cos\varphi_{10}, \qquad \sqrt{5}\sin\varphi_5 = 2\sin 2\varphi_5 - \sin\varphi_5.$$

For the Pell case we find

$$\begin{aligned} 2\cos a\varphi_8 &= \beta_1(a) + \beta_1'(a)\sqrt{2} + \beta_1''(a)\cos\varphi_8 + \beta_1'''(a)\cos 3\varphi_8, \\ \cos(2a-1)\varphi_8 &= \beta_2(a)\cos\varphi_8 + \beta_2'(a)\cos 3\varphi_8, \\ \sin(2a-1)\varphi_8 &= \beta_3(a)\sin\varphi_8 + \beta_3'(a)\sin 3\varphi_8: \end{aligned} \tag{48.1}$$

$$\beta_1(a) = (0_7\ \bar{2}\ 0_7\ 2)_{16}, \quad \beta_1'(a) = (0\ 1\ 0_3\ (\bar{1}\ 0_3)_2\ 1\ 0_2)_{16},$$
$$\beta_1''(a) = (2\ 0_5\ \bar{2}\ 0\ \bar{2}\ 0_5\ 2\ 0)_{16}, \quad \beta_1'''(a) = (0_2\ 2\ 0\ \bar{2}\ 0_5\ \bar{2}\ 0\ 2\ 0_3)_{16},$$
$$\beta_2(a) = (1\ 0_2\ \bar{1}_2\ 0_2\ 1)_8, \quad \beta_2'(a) = (0\ 1\ \bar{1}\ 0_2\ \bar{1}\ 1\ 0)_8,$$
$$\beta_3(a) = (1\ 0_2\ 1\ \bar{1}\ 0_2\ \bar{1})_8, \quad \beta_3'(a) = (0\ 1_2\ 0_2\ \bar{1}_2\ 0)_8.$$

$K(\sqrt{2})$, $K(\cos\varphi_8)$, $K(\cos 3\varphi_8)$ are distinct, also $K(\sin\varphi_8)$, $K(\sin 3\varphi_8)$. For the Lucas case

$$\begin{aligned} 4\cos a\varphi_{12} &= \gamma_1(a) + \gamma_1'(a)\sqrt{2} + \gamma_1''(a)\sqrt{3} + \gamma_1'''(a)\sqrt{6}, \\ 4\cos(2a-1)\varphi_{12} &= \gamma_2(a)\sqrt{2} + \gamma_2'(a)\sqrt{6}, \\ 4\sin(2a-1)\varphi_{12} &= \gamma_3(a)\sqrt{2} + \gamma_3'(a)\sqrt{6}: \end{aligned} \tag{49.1}$$

$$\gamma_1(a) = (0_3\ 2\ 0_3\ \bar{2}\ 0_3\ \bar{4}\ 0_3\ \bar{2}\ 0_3\ 2\ 0_3\ 4)_{24},$$
$$\gamma_1'(a) = (1\ 0\ 2\ 0\ \bar{1}\ 0\ 1\ 0\ \bar{2}\ (0\ \bar{1})_2\ 0\ \bar{2}\ 0\ 1\ 0\ \bar{1}\ 0\ 2\ 0\ 1\ 0)_{24},$$
$$\gamma_1''(a) = (0\ 2\ 0_7\ \bar{2}\ 0_3\ \bar{2}\ 0_7\ 2\ 0_2)_{24},$$
$$\gamma_1'''(a) = (1\ 0_3\ 1\ 0\ \bar{1}\ 0_3\ \bar{1}\ 0\ \bar{1}\ 0_3\ \bar{1}\ 0\ 1\ 0_3\ 1\ 0)_{24},$$
$$\gamma_2(a) = (1\ 2\ \bar{1}\ 1\ \bar{2}\ \bar{1}_2\ \bar{2}\ 1\ \bar{1}\ 2\ 1)_{12}, \quad \gamma_2'(a) = (1\ 0\ 1\ \bar{1}\ 0\ \bar{1}_2\ 0\ \bar{1}\ 1\ 0\ 1)_{12},$$
$$\gamma_3(a) = (\bar{1}\ 2\ \bar{1}_2\ 2\ \bar{1}\ 1\ \bar{2}\ \bar{1}_2\ \bar{2}\ 1)_{12}, \quad \gamma_3'(a) = (1\ 0\ 1_2\ 0\ 1\ \bar{1}\ 0\ \bar{1}_2\ 0\ 1)_{12}.$$

$$(51.1)\quad \cos a\varphi_1 = (-1)^a, \quad \cos(2a-1)\varphi_2 = 0, \quad \sin(2a-1)\varphi_2 = (-1)^{a-1}.$$

For the Fermat and miscellaneous cases

$$2\cos a\varphi_3 = \delta_1(a) = (1\,\bar{1}\,\bar{2}\,\bar{1}\,1\,2)_6,$$
$$2\cos 2a\varphi_3 = \delta_2(a) = (\bar{1}_2\,0)_3,$$
$$2\cos(2a-1)\varphi_3 = \delta_3(a) = (1\,\bar{2}\,1)_3,$$
$$2\sin(2a-1)\varphi_3 = \delta_4(a)\sqrt{3}, \qquad \delta_4(a) = (1\,0\,\bar{1})_3.$$
$$2\cos a\varphi_4 = \varepsilon_1(a) + \varepsilon_1'(a)\sqrt{2}, \quad \varepsilon_1(a) = (0_3\,\bar{2}\,0_2\,\bar{1}\,2)_8, \quad \varepsilon_1'(a) = (1\,0\,\bar{1}\,0\,\bar{1}\,0)_8,$$
$$2\cos(2a-1)\varphi_4 = \varepsilon_2(a)\sqrt{2}, \qquad \varepsilon_2(a) = (1\,\bar{1}_2\,1)_4,$$
$$2\sin(2a-1)\varphi_4 = \varepsilon_3(a)\sqrt{2}, \qquad \varepsilon_3(a) = (1_2\,\bar{1}_2)_4.$$
$$2\cos a\varphi_6 = \zeta_1(a) + \zeta_1'(a)\sqrt{3},$$
$$\zeta_1(a) = (0\,1\,0\,\bar{1}\,0\,\bar{2}\,0\,\bar{1}\,0\,1\,0\,2)_{12}, \qquad \zeta_1'(a) = (1\,0_3\,\bar{1}\,0\,\bar{1}\,0_3\,1\,0)_{12},$$
$$2\cos(2a-1)\varphi_6 = \zeta_2(a)\sqrt{3}, \qquad \zeta_2(a) = (1\,0\,\bar{1}_2\,0\,1)_6,$$
$$2\sin(2a-1)\varphi_6 = \zeta_3(a)\sqrt{3}, \qquad \zeta_3(a) = (1\,2\,1\,\bar{1}\,\bar{2}\,\bar{1})_6.$$

It is possible to state $\alpha, \beta, \cdots, \zeta$ as explicit functions of their arguments by means of Legendre-Jacobi symbols; in all but a few cases the results are too complicated to be of interest.

16. **Fibonacci relations.** These are written out by applying (47), (47.1) (47.2) to the identities in § 5 for $x = \varphi_5$ or φ_{16}. For these values (4.1) gives

$$(\mathrm{XIV})\qquad \begin{aligned} \sum v_r O_{2r}(2n) &= 2P(n) + \sum \alpha_2(a) P(n-2a^2), \\ \sum u_r O_{2r}(2n) &= \sum \alpha_2'(a) P(n-2a^2), \end{aligned}$$
$$\sum 5^r[v_{2r}O_{4r}(2n) + 5u_{2r+1}O_{4r+2}(2n)] = 2P(n) + \sum \alpha_1(a) P(n-2a^2),$$
$$\sum 5^r[u_{2r}O_{4r}(2n) + v_{2r+1}O_{4r+2}(2n)] = \sum \alpha_1'(a) P(n-2a^2).$$

Similarly from (5.1),

$$(\mathrm{XV})\qquad \begin{aligned} \sum v_{2r+1}O_{2r+1}(2n+1) &= \sum \alpha_3(a) P(n-4t_a), \\ \sum u_{2r+1}O_{2r+1}(2n+1) &= \sum \alpha_3'(a) P(n-4t_a), \end{aligned}$$
$$\sum 5^r[u_{2r+1}O_{4r+1}(2n+1) + v_{2r+2}O_{4r+3}(2n+1)] = \sum \alpha_8(a) P(n-4t_a),$$
$$\sum 5^r[v_{2r+1}O_{4r+1}(2n+1) + u_{2r+1}O_{4r+3}(2n+1)] = \sum \alpha_8'(a) P(n-4t_a),$$
$$\alpha_8(a) = (1\,0_3\,\bar{1}_2\,0_3\,1)_{10}, \qquad \alpha_8'(a) = (1\,2\,0\,\bar{2}\,\bar{1}_2\,\bar{2}\,0\,2\,1)_{10}.$$

From (6.1),

$$\sum(-1)^r u_r A_r(n) = \sum \alpha_9(a) P(n-t_a),$$

$$\sum(-1)^r v_r A_r(n) = \sum \alpha_9'(a) P(n-t_a),$$

$$\text{(XVI)} \qquad \alpha_9(a) = (0\,\overline{1}\,2\,\overline{1}\,0)_5, \qquad \alpha_9'(a) = (2\,\overline{3}\,\overline{2}\,\overline{3}\,2)_5,$$

$$\sum u_r A_r(n) = \sum \alpha_6(a) P(n-t_a),$$

$$\sum v_r A_r(n) = \sum \alpha_{10}(a) P(n-t_a),$$

$$\alpha_{10}(a) = (2\,\overline{1}\,0\,\overline{1}\,\overline{2}_2\,1\,0\,\overline{1}\,2)_{10}.$$

From (6.2),

$$5A_0(n) + \sum(2v_a + v_{a+1}) A_a(n) = 5P(n) + 5\sum \alpha_{11}(a) P(n-t_{a+1}),$$

$$\text{(XVII)} \quad A_0(n) + \sum(2u_a + u_{a+1}) A_a(n) = P(n) + \sum \alpha_{11}'(a) P(n-t_{a+1}),$$

$$\alpha_{11}(a) = (0_3\,\overline{1}_2\,0_3\,1_2)_{10}, \qquad \alpha_{11}'(a) = (2\,0_2\,\overline{1}_2\,\overline{2}\,0_2\,1_2)_{10};$$

write for brevity $B_0(n) = A_0(n)$, $B_a(n) = A_{a-1}(n) + 2A_a(n)$;

$$4B_0(n) + \sum 5^a [v_{2a} B_{4a}(n) + u_{2a-1} B_{4a-2}(n)]$$
$$= 4P(n) + \sum \alpha_{12}(a) P(n-t_{a+1}),$$

$$\sum 5^{a-1} [5u_{2a} B_{4a}(n) + v_{2a-1} B_{4a-2}(n)] = \sum \alpha_{12}'(a) P(n-t_{a+1}),$$

$$\sum 5^{a-1} [(v_{2a-1} + 5u_{2a-1}) B_{4a-1} + (v_{2a-2} + u_{2a-2}) B_{4a-8}]$$
$$= 2P(n) + \sum \alpha_{12}''(a) P(n-t_{a+1}),$$

$$\sum 5^{a-1} [v_{2a-1} B_{4a-1} + u_{2a-2} B_{4a-8}] = \sum \alpha_{12}'''(a) P(n-t_{a+1}),$$

$$\alpha_{12}(a) = (1_2\,\overline{1}_2\,1_2\,\overline{4}_2\,\overline{1}_2\,1_2\,\overline{1}_2\,1_2\,4_2)_{20}, \qquad \alpha_{12}'(a) = (1_4\,\overline{1}_4\,0_2\,\overline{1}_4\,1_4\,0_2)_{20},$$

$$\alpha_{12}''(a) = (2\,0_6\,\overline{2}_4\,0_6\,2_3)_{20}, \qquad \alpha_{12}'''(a) = (0\,1_2\,0_2\,\overline{1}_2\,0_4\,\overline{1}_2\,0_2\,1_2\,0_3)_{20}.$$

From (4.11),

$$\sum 5^r [v_{2r} O_{4r}(2n) + 5u_{2r+1} O_{4r+2}(2n)]$$
$$= 2P(n) + \sum(-1)^a \alpha_2(a) P(n-2a^2),$$

$$\text{(XVIII)} \quad \sum 5^r [u_{2r} O_{4r}(2n) + v_{2r+1} O_{4r+2}(2n)]$$
$$= -\sum(-1)^a \alpha_2'(a) P(n-2a^2);$$

$$\sum v_{2r} O_{2r}(2n) = 2P(n) + \sum(-1)^a \alpha_1(a) P(n-2a^2),$$

$$\sum u_{2r} O_{2r}(2n) = -\sum(-1)^a \alpha_1'(a) P(n-2a^2).$$

From (5.11),

$$\begin{aligned}
&\sum 5^r\left[u_{2r}O_{4r+1}(2n+1)+v_{2r+1}O_{4r+3}(2n+1)\right]\\
&\qquad\qquad=\sum(-1)^a\alpha_4'(a)\,P(n-4t_a),\\
&\sum 5^r\left[v_{2r}O_{4r+1}(2n+1)+5u_{2r+1}O_{4r+3}(2n+1)\right]\\
&\qquad\qquad=\sum(-1)^a\alpha_{13}(a)\,P(n-4t_a),\\
&\alpha_{13}(a)=(\bar{1}_2\,0\,1_2)_5;\\
&\sum v_{2r}O_{2r+1}(2n+1)=-\sum(-1)^a\alpha_7(a)\,P(n-4t_a),\\
&\sum u_{2r}O_{2r+1}(2n+1)=\sum(-1)^a\alpha_7'(a)\,P(n-4t_a).
\end{aligned}\tag{XIX}$$

From (6.11),

$$\begin{aligned}
&\sum u_rA_r(n)=2\sum(-1)^a\alpha_4'(a)\,P(n-t_a),\\
&\sum v_rA_r(n)=2\sum(-1)^a\alpha_{14}(a)\,P(n-t_a),\qquad \alpha_{14}(a)=(1\,2\,0\,\bar{2}\,\bar{1})_5;\\
&\sum(-1)^r v_rA_r(n)=\sum(-1)^a\alpha_{15}'(a)\,P(n-t_a),\\
&\sum(-1)^r u_rA_r(n)=\sum(-1)^a\alpha_{15}(a)\,P(n-t_a),\\
&\alpha_{15}(a)=(1\,0\,\bar{2}\,0\,1\,\bar{1}\,0\,2\,0\,\bar{1})_{10},\qquad \alpha_{15}'(a)=(3\,2\,\bar{2}\,2\,3\,\bar{3}\,\bar{2}\,2\,\bar{2}\,\bar{3})_{10}.
\end{aligned}\tag{XX}$$

(6.21) gives nothing new.

17. **Pell, Lucas, Fermat relations.** Similarly to § 16 the Pell relations are written out from (48), (48.1); the Lucas from (49), (49.1) and the Fermat from (50), (51), (51.1). To save space we omit the results.

18. **Miscellaneous and aperiodic relations.** On account of their simplicity we include a few relations found by applying certain of the formulas in (50), (51.1) to the fundamental identities. In (4.1), (5.1) put $x=\varphi_3$, and in (6.1) $x=\varphi_6$:

$$\begin{aligned}
O(2n)&=P(n)+\sum\delta_2(a)\,P(n-2a^2),\\
O(2n+1)&=\sum\delta_3(a)\,P(n-4t_a),\\
A(n)&=\sum\zeta_2(a)\,P(n-t_a),
\end{aligned}\tag{XXI}$$

which express the O, A functions in terms of Ps.

The next are found in a wholly different way from any of the preceding. In the fundamental identities (4)–(6.21) reduce all powers of cosines or sines on the left to functions of multiple angles and in the results equate coefficients of cosines or sines having equal arguments; or express all cosines or sines of multiple angles on the right of the original identities in terms of powers of $\cos x$, $\sin x$, and in these results equate coefficients of like powers of $\cos x$ or $\sin x$.

Equating first coefficients of $\cos 2sx$ then of $\cos^{2s} x$, $s \geqq 0$, in (4.1) we find

$$\text{(XXII)} \quad \begin{aligned} P(n) &= \sum \binom{2s+2r}{r} O_{2s+2r}(2n+4s^2), \\ O_{2s}(2n) &= P(n-2s^2) + 2\sum (-1)^a \frac{s+a}{2s+a} \binom{2s+a}{a} P(n-2(s+a)^2). \end{aligned}$$

The first expresses $P(n)$ linearly in terms of O-partitions of ranks $2s$, $2(s+1)$, $2(s+2)$, $\cdots$ of the fixed integer $2n+4s^2$; the second gives the O-partitions of $2n$ of fixed rank $2s$ in terms of all the partitions of $n-2s^2$, $n-2(s-1)^2$, $n-2(s-2)^2$, $\cdots$. From their derivations these are valid for $s \geqq 0$. Similarly from (5.1),

$$\text{(XXIII)} \quad \begin{aligned} P(n) &= \sum \binom{2s+2r+1}{r} O_{2s+2r+1}(2n+(2s+1)^2), \\ O_{2s+1}(2n+1) &= \sum (-1)^{a-1} \frac{2s+2a-1}{2s+a} \binom{2s+a}{a-1} P(n-4t_{s+a}). \end{aligned}$$

We omit the corresponding pair for $A_a(n)$, $P(n)$ found in the same way from (6.2). By combining Euler's expansion of $\prod (1-x^n)$ with the simplest identities between the theta functions it is easy to derive recurrences for $A(n)$, $O(n)$ in which the arguments decrease by multiples of the successive triangular numbers.

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December, 1921.